Incompressible Boundary Layer Flow

We shall consider two Regions.

1. A very thin Layer in the immediate vicinity of the surface in which $\frac{\partial u}{\partial y}$ is very Large so that even for small μ, $\tau_w = \mu \frac{\partial u}{\partial y}$ will be large.

2. Region outside of the Boundary Layer (BL) when τ is small and can be neglected.

Governing equations

Assuming two-dimensional, incompressible and constant property flow.

Continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Navier-stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(2)

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Defining Reynolds number

$$Re = \frac{\rho V_\infty l}{\mu} \qquad (3)$$

l = characteristic length
V_∞ = free stream velocity

We non-dimensionalize these equations

$$u^* = \frac{u}{V_\infty} \qquad x^* = \frac{x}{l}$$

From the exact solution of the stagnation flow

$$\frac{\delta}{l} = \frac{1}{Re^{1/2}} \quad \text{so let} \quad y^* = \frac{y}{l} = Re^{1/2}\frac{y}{l}$$

From eq. (1)

$$\frac{U_\infty}{l}\frac{\partial u^*}{\partial x^*} + \frac{Re^{1/2}}{l}\frac{\partial v}{\partial y^*} = 0 \qquad (4)$$

In order that this expression be non-dimensional we should define

$$v^* = \frac{R^{1/2} v}{U_\infty} \qquad (5)$$

Equation (2)

$$U_\infty\frac{\partial u^*}{\partial t} + \frac{U_\infty^2}{l}u^*\frac{\partial u^*}{\partial x^*} + \frac{U_\infty^2}{Re^{1/2}}v^*\frac{\partial u^*}{\partial y^*}\cdot\frac{Re^{1/2}}{l} =$$

$$-\frac{1}{\rho l}\frac{\partial p}{\partial x^*} + \nu\frac{U_\infty}{l^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \nu\frac{U_\infty}{l^2}\cdot Re\frac{\partial^2 u^*}{\partial y^{*2}} \qquad (6)$$

If we divide this equation by $\frac{U_\infty^2}{l}$ and

$$t^* = U_\infty\frac{t}{l} \quad \text{and} \quad P^* = \frac{p}{\rho U_\infty^2}$$

$$(7) \quad \frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

As $Re \to \infty$ the only term to be affected is the second from the last, and it, therefore, can be neglected

The same equation for the y-component.

$$(8)\quad \frac{1}{Re}\left(\frac{\partial v^*}{\partial t^*} + \frac{u^*\partial v^*}{\partial x^*} + \frac{v^*\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y} + \frac{1}{Re^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial y^{*2}}$$

As $Re \to \infty$ eq (8) is reduced to

$$\frac{\partial p^*}{\partial y^*} = 0$$ this indicates that $p = p(x)$.

Then the final boundary equation then, are

Continuity

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad (9)$$

Momentum

$$\frac{\partial u^*}{\partial t^*} + \frac{u^*\partial u^*}{\partial x^*} + \frac{v^*\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}} \qquad (10)$$

with boundary conditions

- $u^*(x^*, 0) = 0$
- $v^*(x^*, 0) = V_0$ or 0, where V_0 represents a suction or blowing at wall.
- $u^*(x^*, \infty) = U_\infty(x)$
- At $x^* = 0$ the velocity profile must be specified.
- No explicit restrictions are placed on v^* at either $x^* = 0$ or $y^* \to \infty$

What are the implications of B.L. equations

1 All terms in (10) are of the same order of magnitude.

$$u^* - O(1) \quad \frac{\partial u^*}{\partial x^*} - O(1) \quad v^* O(1) \quad \frac{\partial u^*}{\partial y^*} \sim O(1)$$

2. By neglecting the axial diffussion term the equations are now parabolic, that is, what happens downstream will not affect what happens upstream. This implies the analysis, especially whe numerical techniques are used.

3. The momentum equation has been reduced to a third order equation and, therefore, one less boundary condition can be satisfied. This explain why $v(\infty, 0)$ cannot be satisfied.

4) The pressure in the B.L. depends only on the imposed pressure in the free stream thus the pressure distribution can be determined from potential flow solutions. For external flow $\frac{dP}{dx}$, is known.

5. The Boundary layer thickness is small compared to the characteristic length that is $\frac{\delta}{l} \sim \frac{1}{Re^{1/2}}$

6. Normal velocity is very small compared to the axial velocity. From

$$v^* = \frac{Re^{1/2} v}{U_\infty}$$

as Re becomes very large, v must become small in order that v^* remains of the same order of magnitude as the other terms on (10)

7. As $l \to 0$ Boundary layer equations become invalid. At the leading edge of the surface then, we must seek solutions to the complete Navier-Stokes equations. One of the reasons is that the axial diffusion no longer is negligible.

Boundary Layer Approximation

There are two situations where the viscous terms in the N-S equations can be neglected.

The first occurs in high Reynolds number regions of the flow where net viscous forces are known to be negligible compared to inertial and/or pressure forces, we know this region and it is inviscid region flow. Second situation occurs when the vorticity is negligibly small, we call these irrotational or potential regions of flow. These two situation yields the Euler equation.

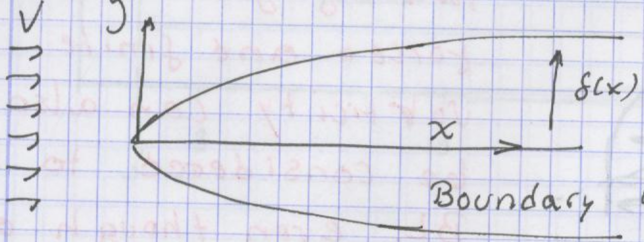
There are serious deficiencies associated with the application of the Euler equation to practical engineering flow problems. One is the inability to specify the no-slip conditions at solid walls. Zero viscous shear forces on solid walls and zero aerodynamic drag on bodies immersed in free stream are unpractical results.

The Boundary layer approximation bridges the gap between the Euler equation and Navier-Stoke equation; and between the slip condition and the no-slip condition at solid walls.

Ludwig Prandtl (1875-1953) introduced the boundary layer approximation. The idea was to divide the flow into regions, outer flow region and irrotational and inner flow region called a boundary layer - a very thin region of flow near a solid wall where viscous forces and rotationality cannot be ignored.

outer flow (inviscid and/or irrotational region flow)

- Euler equation
- Continuity equation } to obtain velocity flow outer flow
- Bernoulli equation } to obtain pressure field.



Boundary layer (rotational with non negligible viscous forces)

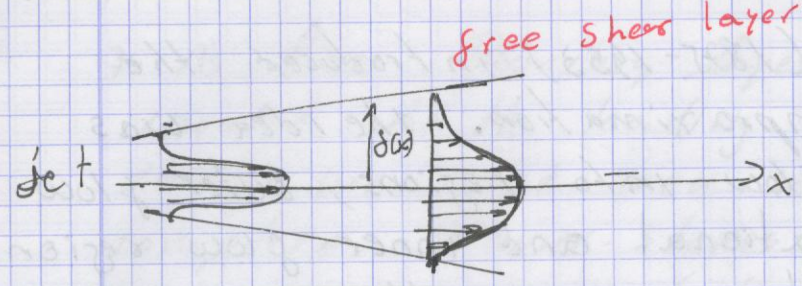
Boundary layer equation

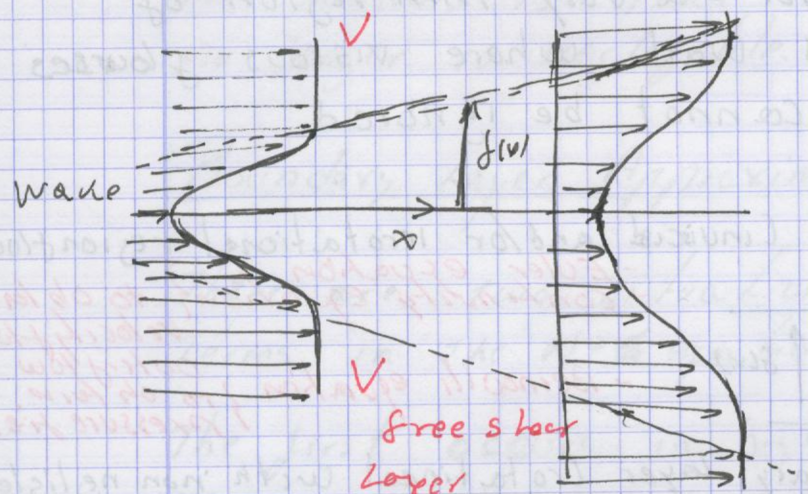
Boundary layer equation corrects some of the major deficiencies of the Euler equation by providing a way to enforce the ~~noo~~-slip condition at solid walls.

Note: Today, therefore, it's no longer necessary to split the flow into outer flow regions and boundary layer regions - we can use CFD to solve the full set of equations of motion throughout the whole flow field.

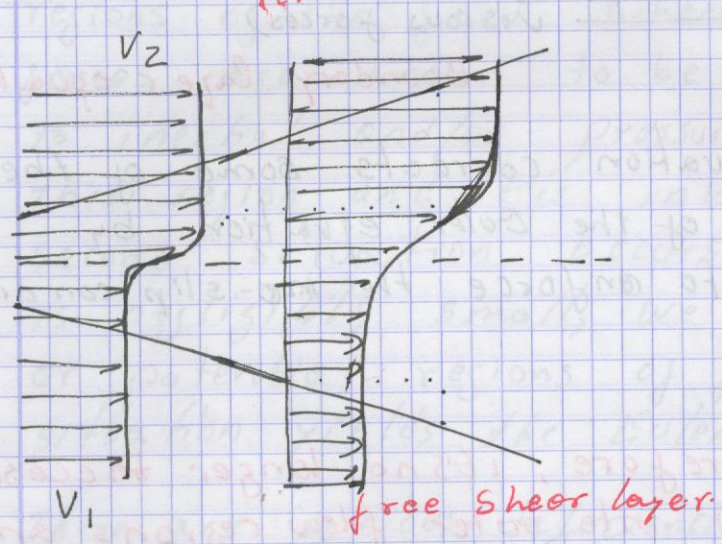
The key to successful application of the boundary layer approximation is the assumption that the boundary layer is very thin.

Boundary layers approximation is not limited to wall-bounded flow regions. The same equations can be applied to free shear layers such as jet, wakes and mixing layers



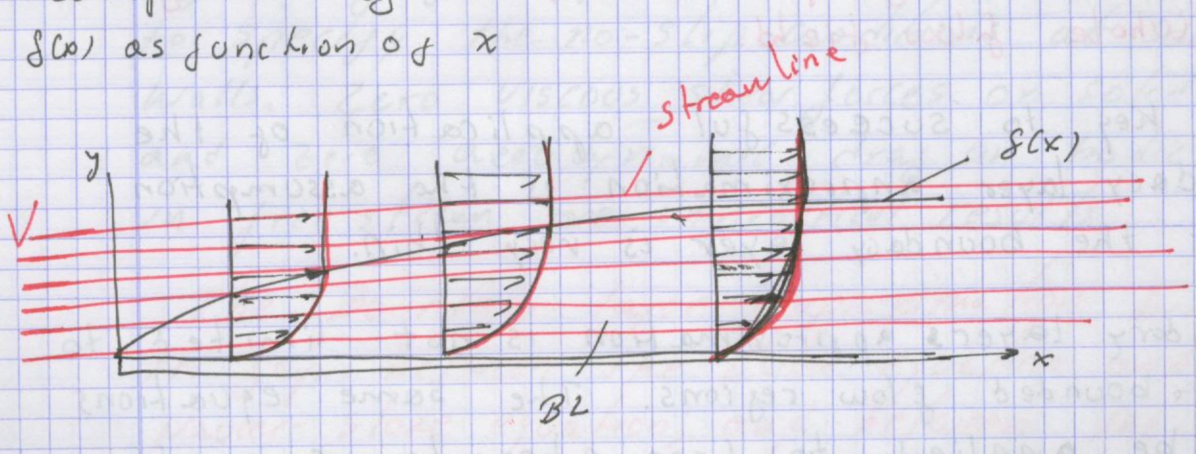


High Reynolds number sufficiently to assume very thin region. The Region of these flow fields with non-negligible viscous forces and finite vorticity can also be considered to be BL. even though a solid wall boundary may not even be present

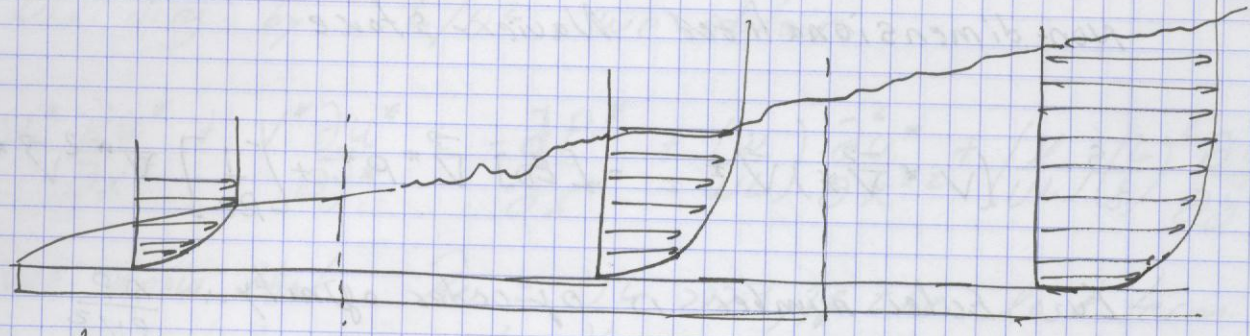


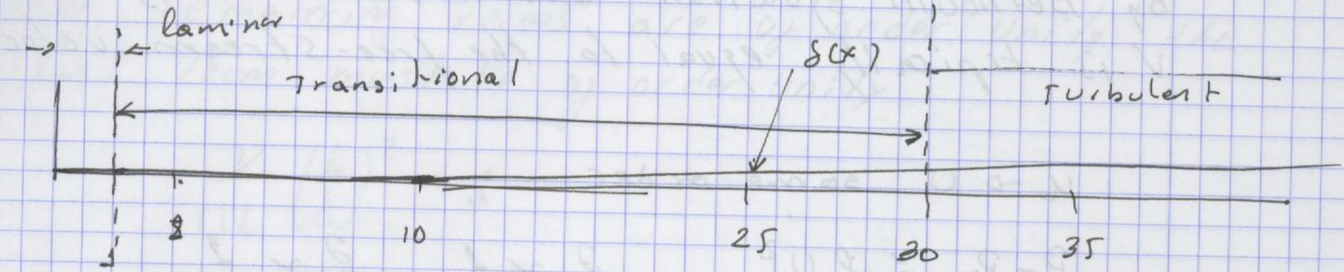
$\delta(x)$ ~~BL~~ thickness where the velocity is 99 percent of the maximum. δ is not a streamline.

Comparison of streamlines and the curve representing $\delta(x)$ as function of x



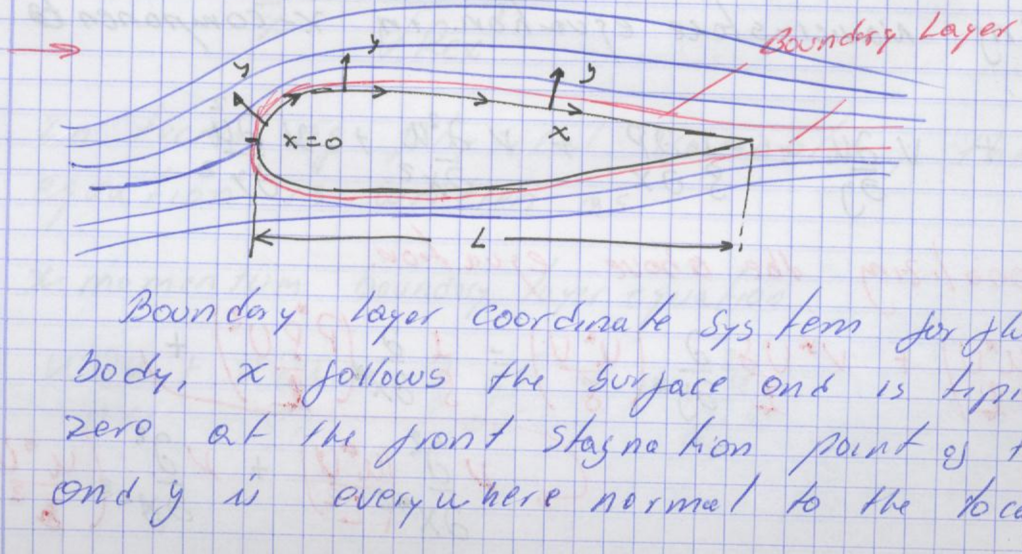
For laminar boundary layer growing on a flat plate, δ, depends on V, x and fluid properties ρ and μ. Re_x, critical, is around 1×10^5 and 3×10^6 Transition Reynolds number





Boundary Layer Equations

For simplicity we consider only steady, Two-dimension flow xy-plane in cartesian coordinates. This can be extended, however, to axisymmetric boundary layer or to three-dimensional boundary layer in any coordinate system. We neglect gravity, we are no dealing with free surfaces or with buoyancy-driven flows (free-convection flow) where gravity effect dominate.



Boundary layer coordinate system for flow over a body, x follows the surface and is typically set to zero at the front stagnation point of the body and y is everywhere normal to the locally surface

Non-dimensionalized Navier-stoke

S $(V^* \vec{\nabla}^*)\vec{V}^* = -[Eu]\,\vec{\nabla}^* P^* + \left[\frac{1}{Re}\right]\nabla^{*2}\vec{V}^*$

Eu: Euler number is of order of unity.: $\frac{\Delta P}{\rho V^2}$

Pressure differences outside the BL. are determined by Bernoulli equation and $\Delta P = P - P_\infty \approx \rho V^2$

V is tipically equal to the free-stream velocity

$u \to U$ same order

$P - P_\infty \cong \rho U^2 \qquad \frac{\partial}{\partial x} \sim \frac{1}{L} \qquad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$

Incompressible continuity

$$\underbrace{\frac{\partial u}{\partial x}}_{U/L} + \underbrace{\frac{\partial v}{\partial y}}_{V/\delta} = 0 \quad \Rightarrow \quad \frac{U}{L} + \frac{V}{\delta} = 0$$

$$\frac{U}{L} \cong \frac{V}{\delta}$$

The magnitude of velocity component v, is given by

$V \cong \frac{U\delta}{L}$ and we know $\frac{\delta}{L} \lll 1$ we can conclude $v \lll u$ in boundary layer

Applaying Navier-stoke equation in x-components

$$u.\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

Dimensionalizing the above equation.

$$u^* U \frac{\partial}{\partial x^*}\left(\frac{u^* U}{L}\right) + v^* \frac{U\delta}{L}\frac{\partial}{\partial y^*}\left(\frac{u^* U}{\delta}\right) = \frac{1}{\rho}\frac{\partial}{\partial x^*}\left(\frac{P^* \rho U^2}{L}\right) +$$

$$\nu\frac{\partial^2}{\partial x^{*2}}\left(\frac{u^* U}{L^2}\right) + \nu\frac{\partial^2}{\partial y^{*2}}\left(\frac{u^* U}{\delta^2}\right)$$

Dividing by L/U^2 we obtain

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \left(\frac{\nu}{UL}\right) \frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \frac{\partial^2 u^*}{\partial y^{*2}}$$

we know that $Re_L = \frac{UL}{\nu} \ggg 1$, If the last term is neglected, we obtain the Euler equation; clearly this term must remain. Furthermore, since all the remaining terms are of order unity the last term must be of order unity

$$\frac{\nu}{UL}\left(\frac{L}{\delta}\right)^2 \sim 1$$

$$\frac{L^2}{\delta^2} \sim \frac{UL}{\nu} \Rightarrow \frac{L}{\delta} = \left(\frac{UL}{\nu}\right)^{1/2}$$

$$\frac{L}{\delta} = \left(\frac{UL}{\nu}\right)^{1/2} \Rightarrow \frac{L}{\delta} = \sqrt{Re_L}$$

$$\frac{\delta}{L} = \frac{1}{\sqrt{Re_L}}$$: This term comfirms that at a given streamwise location along the wall, the larger Reynolds number, the thinner B.L.

If we substitute x for L; we obtain the laminar BL. as a function of the location, δ grows like $\sqrt{x}$

$$\frac{\delta(x)}{x} = \frac{1}{\sqrt{Re_x}}$$

In terms of physical variables; the above equation is written as

x-momentum boundary layer equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

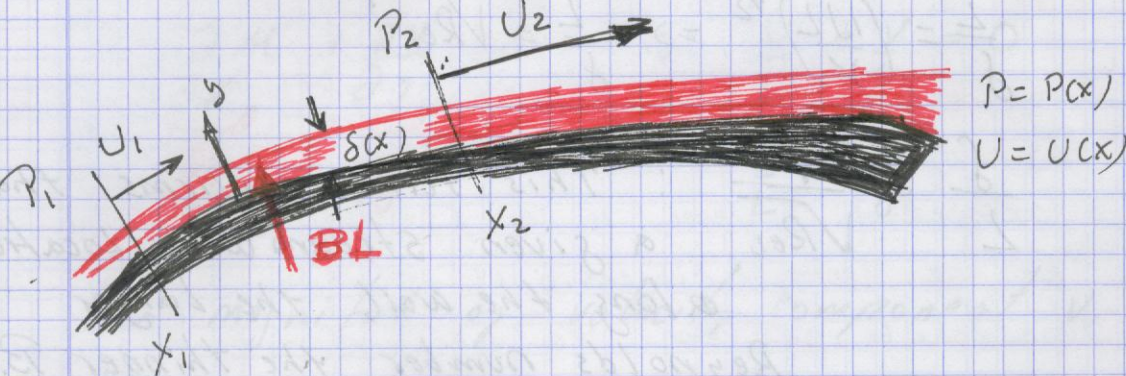
Finally, since we know from our-y-momentum equation analysis that the pressure across the boundary layer is the same as outside the boundary layer, Applaying Bernoulli equation to the outer flow region. Differentiating with respect to x we get

$$\frac{P}{\rho} + \frac{1}{2}U^2 = \text{constant}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = -U\frac{dU}{dx}$$, pluging this equation

in the B.L. equation we obtain

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$



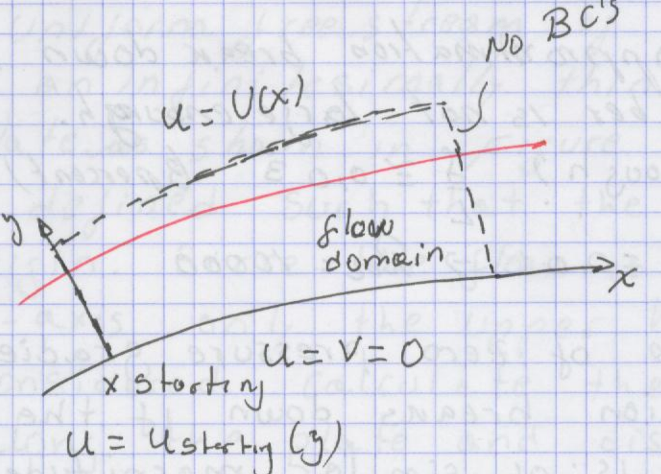
Outer flow speed parallel to the wall is U(x) and is obtained from the outer flow pressure, P(x). This speed appears in the x-component of the boundary layer momentum equation.

Sumarizing the B.L equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$ - Incompresible flow

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$ steady state

Boundary conditions



Boundary layer equation is parabolic, only we need three boundary conditions

$u = v = 0$ at $y = 0$

$u = U(x)$ as $y \to \infty$

$u = u_{starting}(y)$ at $x = x_{starting}$

Boundary Layer Procedure

Step 1: Solve for outer flow, ignoring the Boundary layer (assuming flow outside the BL is approximately inviscid and/or irrotational. Transform coordinates as necessary to obtain $U(x)$.

Step 2: Assume a thin B.L., it doesn't affect the outer flow solution of step 1

Step 3: Solve B.L. equations, using appropiate boundary conditions: No-slip BC's at the wall, $u = v = 0$ at $y = 0$, the known outer flow condition at the edge of the BL $u \to U(x)$ as $y \to \infty$, and some starting profile $u = u_{starting}(y)$ at $x = x_{starting}$

Step 4: Calculate quantities of interest in the flow field. For example once the boundary layer equations have been solve (step 3), we can calculate $\delta(x)$, shear stress, total skin friction drag, etc

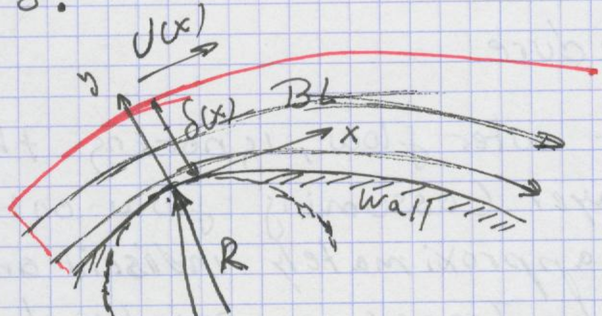
Step 5: Verify that the boundary layer approximations are appropriate. Thus the boundary layer is thin - otherwise the approximation is not justified.

Limitation of the Boundary Layer approximation

- Boundary Layer approximation break down if the Reynolds number is not large enough. How large is enough? $\frac{\delta}{L} \cong 0{,}03$ (3 percent)

$Re_L = 1000$ $\quad \delta/L = 0.01 \rightarrow Re_L = 10000$

- The assumption of zero pressure gradient in the y-direction breaks down if the wall curvature is of similar magnitude as δ.

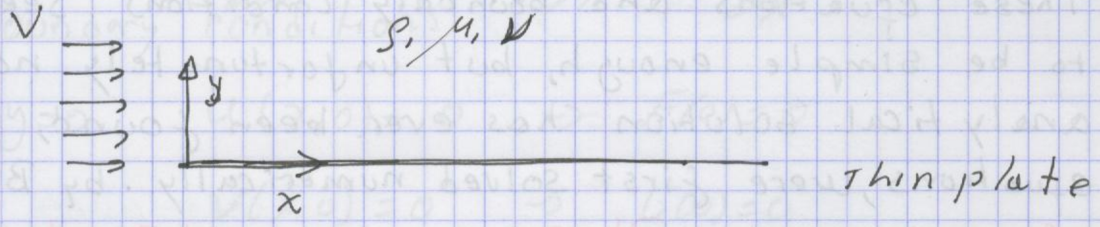


In this case, centripetal acceleration effects due to streamline curvature cannot be ignored. Physically the boundary layer is not thin enough for the approximation to be appropriate when δ is not $\ll R$.

- When Reynolds number is too high, the boundary layer does not remain laminar. The laminar boundary layer on a smooth flat plate under clean flow conditions begins to transition toward turbulence at $Re_x = 1 \times 10^5$. Engineering applications, wall may not be smooth and there may be vibrations, noise and fluctuations in the free-stream flow above wall, all of which contribute to a even earlier start of the transition process.
- If flow separation occurs, the BL. approximation is no longer appropiate in the separated flow region. The main reason for this is that a separated flow region contains reverse flow and the parabolic nature of BL. equations is lost.

Example

A uniform free stream of speed V flow parallel to an infinitesimally thin semi-infinite flat plate as shown in figure. The coordinate system is defined such that the plate begins at the origin. Since the flow is symmetric about x-axis, only the upper half of the flow is considered. Calculate the B.L. velocity profile along the plate and discuss.



Solution.

We need to calculate u as a function of x and y

- Steady state
- Incompressible flow
- Two-dimensional
- Reynolds number is high
- B.L. remains laminar over the range of interest.

Step 1. Calculate $U(x)$ (outer flow)

$$U(x) = V = \text{constant}$$

Step 2. Assume a thin boundary layer

The B.L. is so thin that it has negligible effect on the outer flow calculate in step 1.

Step 3. Solve B.L. equations.

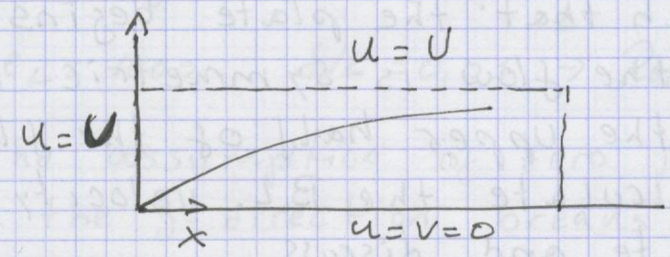
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\overbrace{\frac{dU}{dx}}^{=0} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

Boundary conditions

$u=0$ at $y=0$ $\qquad u=U$ as $y \to \infty$

$v=0$ at $y=0$ $\qquad u=U$ for all y at $x=0$



These equations and boundary conditions seem to be simple enough, but unfortunately no analytical solution has ever been found, however equations, were first solved numerically . by Blasius

As a side Note, Blasius was a phD student of Prandtl.

The key to the solution is the assumption of similarity.

Blasius introduced a similarity variable η. that combines independent variables x and y into one nondimensional ~~independepent~~ independent variable

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and he solved for nondimensionalized form of the x-component of velocity

$$F' = \frac{u}{U} = \text{function of } \eta$$

$$u = U\,F(\eta) \qquad v = \sqrt{\frac{U\nu}{x}}\cdot G(\eta). \qquad (3)$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{U}{\nu x}}\right) = \frac{-\eta}{2x} = -\frac{1}{2}\,y\sqrt{\frac{U}{\nu x}}\,\frac{1}{x} \qquad (4)$$

$$\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left(y\sqrt{\frac{U}{\nu x}}\right) = \frac{\eta}{y} = \sqrt{\frac{U}{\nu x}} \qquad (5)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(U F(\eta)) = U.\ F'(\eta) = U \frac{\partial F(\eta)}{\partial \eta} . \frac{\partial \eta}{\partial x} = -U \frac{\eta}{2x} . F'(\eta) \quad (6)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{\frac{U\nu}{x}}\ G(\eta)\right) = \frac{\partial G(\eta)}{\partial y} = \frac{\partial G}{\partial \eta} . \frac{\partial \eta}{\partial y} = G'(\eta) . \sqrt{\frac{U}{\nu x}} . \sqrt{\frac{U\nu}{x}} \quad (7)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(U F(\eta)\right) = U . \frac{\partial F(\eta)}{\partial y} = U \frac{\partial F}{\partial \eta} . \frac{\partial \eta}{\partial y} = U \sqrt{\frac{U}{\nu x}} . F'(\eta) \quad (8)$$

$$\frac{\partial^2 u}{\partial y^2} = U . \sqrt{\frac{U}{\nu x}}\ F''(\eta) . \sqrt{\frac{U}{\nu x}} = \frac{U}{\nu x}^2 F''(\eta) \quad (9)$$

Boundary conditions

$y=0 \qquad u(x,0) = 0 \quad \rightarrow \quad F(0) = 0$

$\qquad\qquad v(x,0) = 0 \quad \rightarrow \quad G(0) = 0$

$y \rightarrow \infty \qquad u(x,\infty) = U \quad \rightarrow \quad F(\infty) = 1$

continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$-\frac{U}{2x}\eta . F'(\eta) + \frac{U}{x} G'(\eta) = 0$$

$$G'(\eta) = \frac{F'(\eta)\eta}{2} \quad * \quad (10)$$

Momentum equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (11)$$

$$U . F(\eta) . \left(-U \frac{\eta}{2x} F'(\eta)\right) + \sqrt{\frac{U\nu}{x}} . G(\eta) . \left(U \sqrt{\frac{U}{\nu x}}\right) F'(\eta) = \frac{\nu}{\nu} . \frac{U}{x}^2 F''$$

$$\frac{U^2}{x}\left[-F.F'\frac{\eta}{2} + G.F'\right] = \frac{U}{x}^2 F'' \quad (12)$$

$$-\frac{F F' \eta}{2} + G F' = F'' \quad ** \quad (13)$$

We can set up the streamfunction as

$$F = \frac{df}{d\eta} \qquad F' = \frac{d^2 f}{d\eta^2}$$

$$-\frac{1}{2}\frac{df}{d\eta}\cdot\frac{d^2 f}{\partial\eta^2}\cdot\eta + \overbrace{\left(\frac{1}{2}\frac{d f}{d\eta}\eta - f\right)}^{G}\frac{d^2 f}{\partial\eta^2} = \frac{d^3 f}{\partial\eta^3} \quad (14)$$

$$-\frac{1}{2} f \cdot f'' = f'''$$

$$(15) \quad f''' \pm \frac{1}{2} f f'' = 0$$

Runge-Kutta scheme

From $*$ (10)

$$\sigma' = \frac{\eta}{2} F' \Rightarrow \int \frac{\partial\sigma}{\partial\eta}\cdot\partial\eta = \int \frac{\eta}{2}\cdot\frac{\partial F}{\partial\eta}\cdot\partial\eta$$

$$\int \frac{\partial\sigma}{\partial\eta}\partial\eta = \int \frac{\eta}{2}\cdot\frac{\partial^2 f}{\partial\eta^2}\cdot\partial\eta$$

The right side term is solving by integrating by parts

$$\int u\,dv = uv - \int v\,du$$

$$u = \frac{\eta}{2} \qquad du = \frac{d\eta}{2}$$

$$dv = \frac{\partial^2 f}{\partial\eta^2} \qquad v = \frac{\partial f}{\partial\eta}$$

$$\sigma = \frac{\eta}{2} \cdot \frac{\partial f}{\partial \eta} - \int \frac{1}{2} \frac{df}{d\eta} \cdot d\eta$$

$$\sigma = \frac{\eta}{2} \frac{\partial f}{\partial \eta} - \frac{1}{2} f$$

$$\sigma = \frac{1}{2}\left(\eta \frac{\partial f}{\partial \eta} - f\right) \quad (16)$$

Equation.

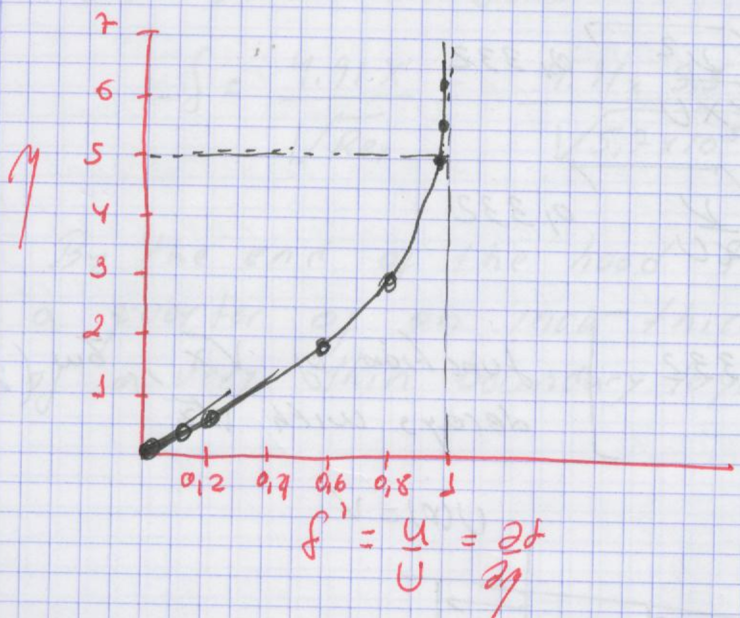
$$\frac{\partial^3 f}{\partial \eta^3} + \frac{f}{2}\frac{\partial^2 f}{\partial \eta^2} = 0 \quad (17)$$

Boundary conditions

$$F(0) = 0 \quad \frac{df}{d\eta} = 0 \quad \text{at} \quad \eta = 0$$

$$F(\infty) = 1 \quad \frac{\partial f}{\partial \eta} = 1 \quad \text{at} \quad \eta = \infty \quad (18)$$

$$\sigma(0) = 0 \quad f = 0 \quad \text{at} \quad \eta = 0$$



Step 4 Calculate quantities of interest

at $\frac{u}{U} = 0{,}99$ the boundary layer thickness from the graph is approx. at $\eta = 4{,}91$

$$\eta = 4.91 = \sqrt{\frac{U}{\nu x}}\,\delta \rightarrow \frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}} \text{ or } \frac{5}{\sqrt{Re_x}} \quad (19)$$

Another quantity of interest is the shear stress at wall τ_w

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \quad (20)$$

In the figure is the slope of nondimensional velocity profile. the nondimensional slope at the wall is

$$\left.\frac{d(u/U)}{d\eta}\right|_{\eta=0} = f''(0) = 0.332$$

$$\tau_w = \mu \cdot U \cdot \sqrt{\frac{U}{\nu x}} \cdot 0.332$$

dividing by U and multiplying U. the ~~out squared~~ squared root

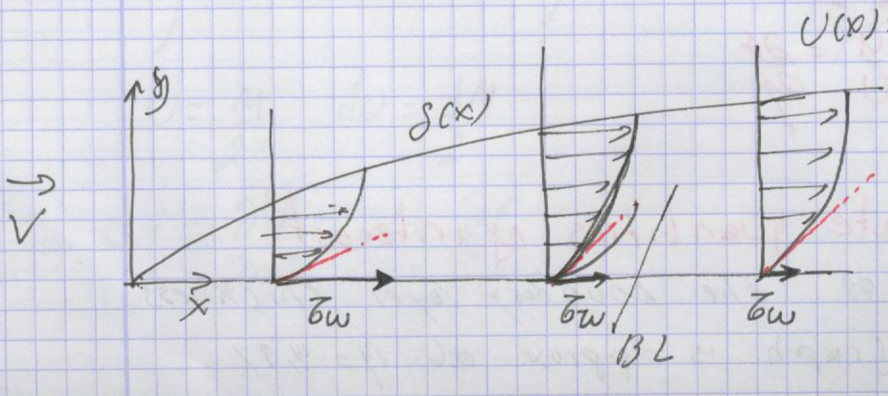
$$\tau_w = \rho \nu U \sqrt{\frac{U}{\nu x}\left(\frac{U}{U}\right)}\, 0.332$$

$$\tau_w = \rho \nu\, U^2 \sqrt{\frac{1}{\nu x U}}\, 0.332$$

$$\tau_w = \rho U^2 \sqrt{\frac{\nu^2}{\nu x U}}\, 0.332$$

$$\tau_w = \rho U^2 \sqrt{\frac{\nu}{x U}}\, 0.332$$

$$\left[\, \tau_w = \frac{\rho U^2}{\sqrt{Re_x}} \cdot 0.332 \,\right]$$ function of $\sqrt{x}$, τ_w decays with $\sqrt{x}$ (21)



Skin friction coefficient

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0,664}{\sqrt{Re_x}} = \frac{\left(\frac{F_{shear\ force}}{A}\right)^{\tau_w}}{\frac{1}{2}\rho U^2} \quad (22)$$

Also decays like inverse of the square root of Reynolds number.

Step 5 Verify that boundary layer is thin

Considering a practical example of flow over the hood of a car while driving at 20mi/h on a hot day. Kinematic viscosity of air is $\nu = 1,8 \times 10^{-4}\ ft^2/s$. Assuming the hood a flat plate of length of 3,5 ft. moving horizontally at a speed of $V = 20mi/h$.

$$Re_x = \frac{Vx}{\nu} = \frac{20\,(mi/h).(3,5ft)}{1,8\times10^{-4}\ ft^2/s}\left(\frac{5280\,ft}{mi}\right)\left(\frac{h}{3600\,ft}\right) = 5,7\times10^5$$

Using the assumption of laminar flow may or may not be appropiate.

We estimate $\delta(x)$.

$$\delta = \frac{4.91x}{\sqrt{Re_x}} = \frac{4.91 \times 3,5\,ft}{\sqrt{5,7\times10^5}} \times \left(\frac{12in}{ft}\right) = 0,27in \cong 0,1cm$$

By the end of the hood the B.L. is only about a quarter of an inch thickness. and our assumption of a very thin boundary layer is verified.

Displacement thickness.

Displacement thickness δ^* is defined as the distance that the streamline just outside of the boundary layer is deflected.

Applying to the control volume and analysis using conservation of mass. We obtain

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \quad (23)$$

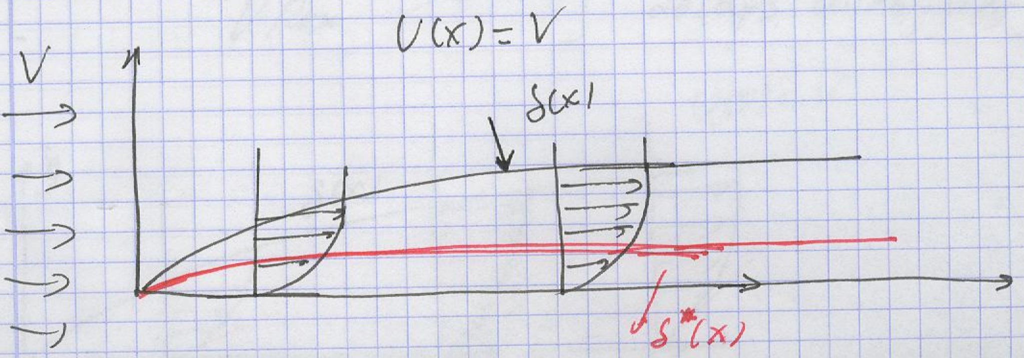
Limit of the integral is ∞, that $y \to \infty$ $u = U$ $\delta^* \to 0$, only we integrate only out some finite distance above δ.

For laminar plate, we integrate the numerical Blasius solution.

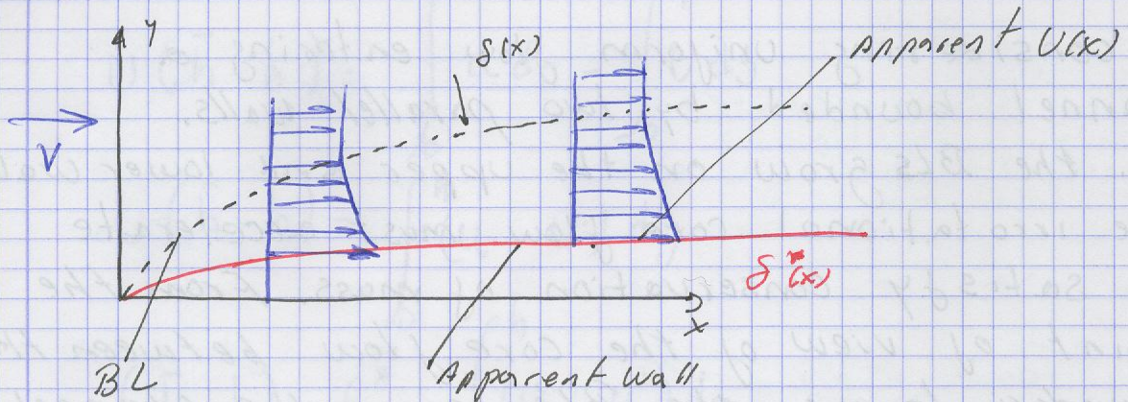
$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}} \quad (24)$$

We see that δ^* is three times smaller than δ at that same x-location.

Engineering meaning of δ^* is that δ^* can be seen as an imaginary or apparent increase in thickness of the wall from the point of view of the inviscid and or irrotational outer region.



Laminar flate plate BL, the displacement thickness is roughly one-third of 99 percent boundary layer thickness.



BL. affects the outer flow in such a way that the wall appears to take the shape of the displacement thickness. The apparent U(x) differs from the original approximation.

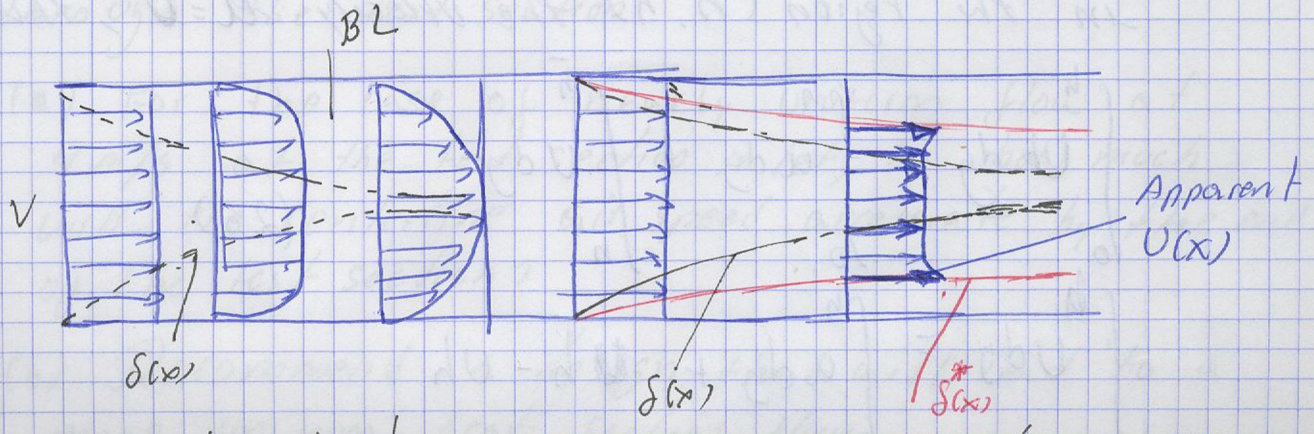
If we were to solve the Euler equation for the flow around this imaginary thicker plate, outer flow velocity component U(x) would differ from the original calculations.

+ We could use this apparent U(x) to impove BL analysis
+ we need to go through the first four steps. calculate $\delta^*(x)$. and then go back to step 1, this time using the imaginary (Thicker) body shape to calculate an apparent U(x).

* Solving the BL. equations again.

* We could repeat the loop as many times as necessary until convergence. In this way, the outer flow and the boundary layer would be more consistent with each other.

If we have a flow in a channel.

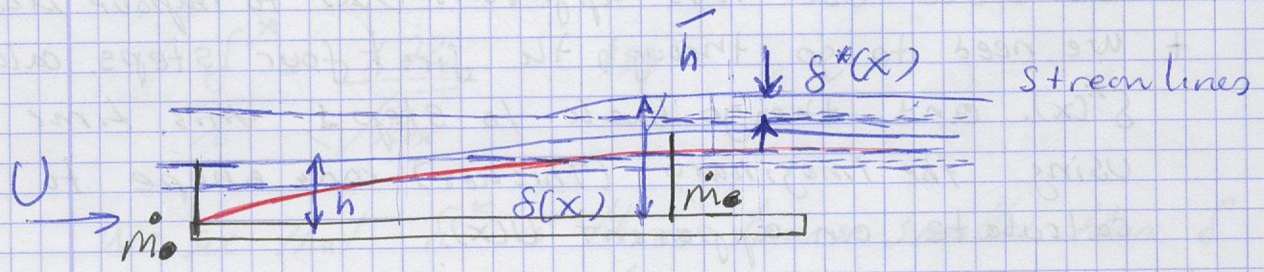


Actual velocity profile

Change in apparent core flow due to the displacement thickness

If considering uniform flow entering a channel bounded by two parallel walls. As the BLs grow on the upper and lower walls, the irrotational core flow must accelerate to satisfy conservation of mass. From the point of view of the core flow between the boundary layers, the BL's cause the channel walls to appear to converge - the apparent distance between the walls decreases as x increases.

The imaginary increase in thickness of one of the walls is equal to $\delta^*(x)$ and the apparent $U(x)$ of the core flow must increase, accordingly as ~~as~~ below figure, to satisfy conservation of mass.



continuity equation

$$\int_0^h U\,dy = \int_0^{\bar{h}} u\,dy = \int_0^h u\,dy + \int_h^{\bar{h}} u\,dy \qquad (25)$$

in the region $(\bar{h}, h)$ the velocity $u = U$, then

$$\int_0^h U\,dy = \int_0^h u\,dy + \int_h^{\bar{h}} U\,dy \qquad (26)$$

$$\int_0^h U\,dy = \int_0^h u\,dy + U\bar{h} - Uh$$

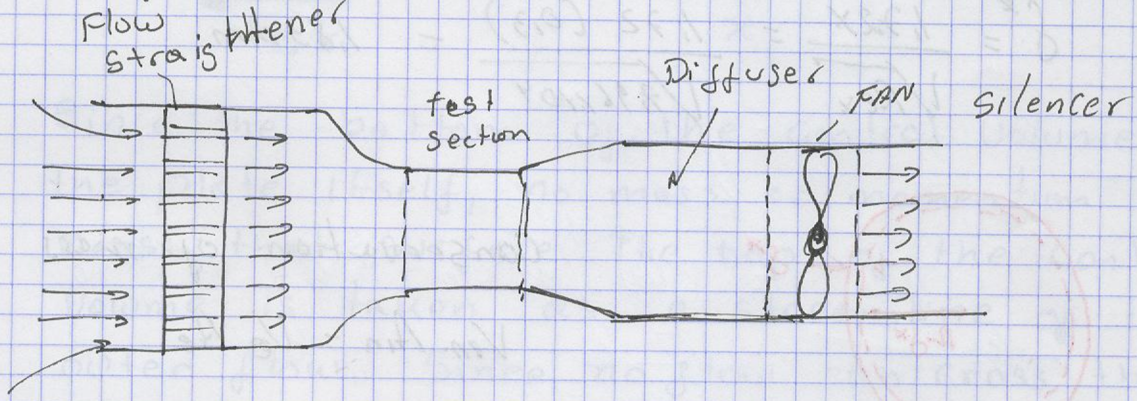
$$U(\bar{h}-h) = \int_0^h u\,dy - \int_0^h U\,dy$$

$$U(\bar{h}-h) = \int_0^h (u-U)\,dy$$

$$(\bar{h}-h) = \delta^* = \int_0^h \frac{(U-u)}{U}\,dy$$

$$\left[\delta^* = \int_0^h \left(1-\frac{u}{U}\right)dy \Rightarrow \int_0^\infty \left(1-\frac{u}{U}\right)dy\right] \quad (27)$$

Example.



The above wind tunnel is being designed for calibration of hot wires. The air is at 19°C. The test section of the wind tunnel is 30 cm in diameter and 30 cm ~~in~~ length. The flow is uniform in the test section. The wind tunnel speed ranges from 1 to 8 m/s. and the design is to optimized for an air speed of 4 m/s through the test section.

(a) For the case of nearly uniform flow at 4 m/s at the test section inlet, by how much will the centerline air speed accelerate by the end of the test section?

(b). Recommend a design that will lead to a more uniform test section flow.

Assumption

- Flow steady state
- Incompressible
- Walls are smooth
- BL. Laminar

We calculate Re_x.

$$Re_x = \frac{\rho \cdot Vx}{\mu} = \frac{Vx}{\nu}$$

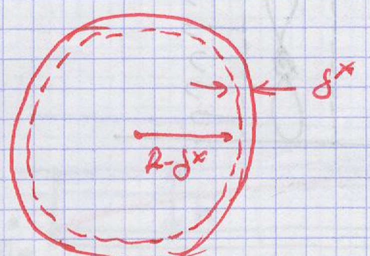
at 19°C $\nu = 1{,}507 \times 10^{-5} \frac{m^2}{s}$

$$Re_L = \frac{VL}{\nu} = \frac{4(m/s)(0{,}3m)}{1{,}507 \times 10^{-5} \frac{m^2}{s}} = 7{,}96 \times 10^4$$

The flow is consider Laminar

We calculate the displacement of the BL. and for conservation of mass we obtain

$$\delta^* = \frac{1{,}72x}{\sqrt{Re_x}} = \frac{1{,}72\,(0{,}3)}{\sqrt{7{,}96 \times 10^4}} = 1{,}83\,mm$$



Test section Cross sectional view. (end at the test section)

Conservation of mass

$$V_{in} \cdot A_{in} = V_e A_e$$

$$V_e = \frac{V_{in} A_{in}}{A_e}$$

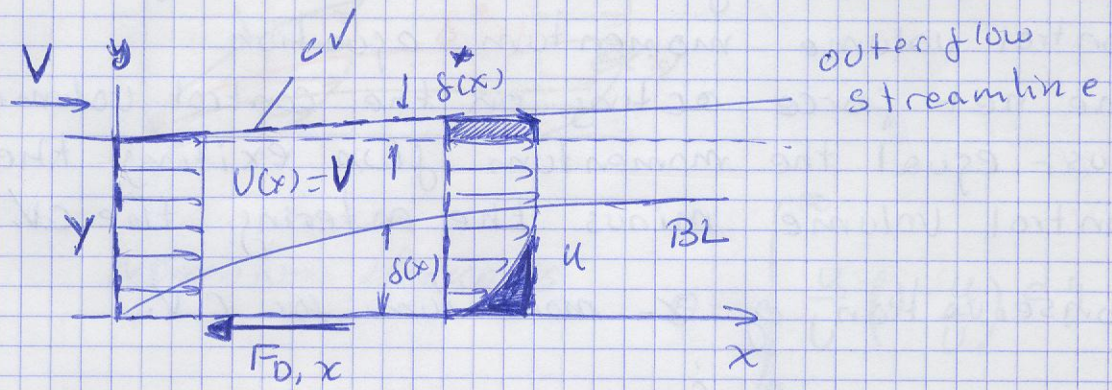
$$V_e = 4. \times \frac{\frac{\pi R^2}{2}}{\frac{\pi (R-\delta^*)^2}{2}}$$

$$V_e = 4. \frac{(0{,}15)^2}{(0{,}15 - 0{,}00183)^2}$$

$$V_e = 4{,}10\ m/s$$

Momentum thickness

It is another measure of boundary layer thickness, commonly give the symbol θ,



Since the bottom of the control volume is the plate itself, no mass or momentum can cross that surface. The top of the control volume is taken as a streamline of the outer flow. Since no flow can cross the streamline, there can be no mass or momentum flux across the upper surface of the C.V.

When applying conservation of mass to this CV.

$$0 = \int_{CS} \rho \vec{V} \vec{n}\, dA = \underbrace{w\rho \int_0^{Y+\delta^*} u\,dy}_{\text{at location } x} - \underbrace{w\rho \int_0^{Y} U\,dy}_{\text{at } x=0} \quad (28)$$

W is the width into the page in the figure

The above intergral is reduced to

$$\int_0^{Y} (U-u)\,dy = U\delta^* \quad (29)$$

Physically, the mass flow deficit within the B.L. (lower blue-shaded) is replaced by a chunk of free-stream of thickness δ^* (The upper blue-region). The above integral shows that the Those shaded regions have the same area.

Now considering the x-component of the control volume momentum equation.
The net force acting on the control volume must equal the momentum flux exiting the control volume minus the entering the C.V.

Conservation of x-momentum for C.V.

$$\sum F_x = -\sum F_{Dx} = \int_{CS} \rho u \vec{V}.\vec{n}\, dA \qquad (30)$$

$$\int_{CS} \rho u \vec{V}\vec{n}\, dA = \rho w \int_0^{Y+\delta^*} u^2\, dy - \rho w \int_0^{Y} U.U\, dy \qquad (31)$$

$-F_{Dx}$: drag force due to the friction

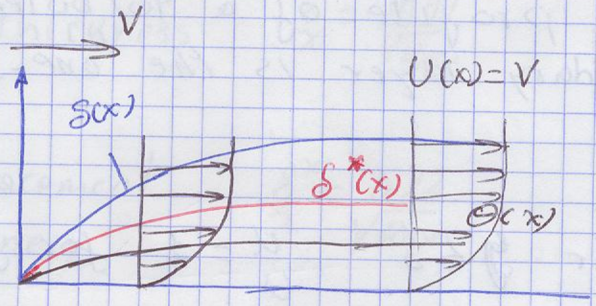
$$F_{Dx} = \rho w \int_0^{Y} u(U-u)\, dy \qquad (32)$$

Finally we define momentum thickness θ such that the viscous drag force on the plate per unit width into the page is equal to ρU^2 times θ.

$$\frac{F_{Dx}}{W} = \rho \int_0^{Y} u(U-u)\, dy = \rho U^2 \theta$$

$$\theta = \int_0^{Y} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad (33)$$

"Momentum thickness is define as the loss of momentum flux per unit width divide by ρU^2 due to the presence of the growing boundary layer"



Momentum thickness $\Theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right) dy$ (34)

For the specific case of Blasios solution for a laminar flat plate B.L, we intergrate the above equation (34) numerically to obtain.

Momentum thickness, Laminar flat plate $\frac{\Theta}{x} = \frac{0,664}{\sqrt{Re_x}}$ (34)

In fact, for laminar flow over a flat plate Θ turns out to be approximately 13.5 percent of δ at any location.

Turbolent Flat Plate Boondary layer

All turbolent expressions discossed in this section repsesent Time-averaged values.
One common empirical approximation for time-averaged velocity profile of a turbolent flat plate boundary layer is the one-seventh power law.

$$\frac{u}{U}=\left(\frac{y}{\delta}\right)^{1/7} \quad \text{for } y \leq \delta \qquad \frac{u}{U}=1 \quad \text{for } y > \delta$$

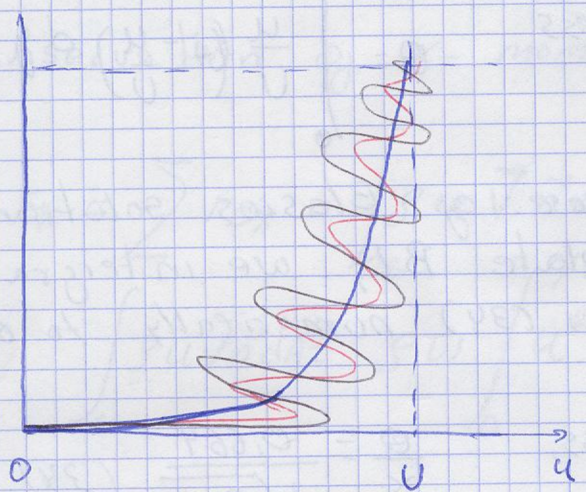


Illustration of the unsteadiness of turbulent BL.; the thin, vary black and red lines are instantaneous profiles and the blue line is a long-time-averaged profile.

The approximate turbolent boundary layer velocity profile is not physically meaningful very close to the wall ($y \to 0$) since it predicts that the slope $\left(\frac{\partial u}{\partial y}\right)$ is infinite at $y=0$.
This large slope at the wall leads to a very high wall shear stress $\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$, and produces high skin friction along the surface of the plate.

Summary of expressions for laminar and Turbulent boundary layers on a smooth flat plate aligned parallel to a uniform stream.

Property	Laminar	Turbulent (a)	Turbulent (b)
B.L. Thickness	$\frac{\delta}{x} = \frac{4,91}{\sqrt{Re_x}}$	$\frac{\delta}{x} \simeq \frac{0,16}{(Re_x)^{1/7}}$	$\frac{\delta^*}{x} \simeq \frac{0,38}{(Re_x)^{1/5}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1,72}{\sqrt{Re_x}}$	$\frac{\delta^*}{x} \simeq \frac{0,020}{(Re_x)^{1/7}}$	$\frac{\delta^*}{x} \simeq \frac{0,048}{(Re_x)^{1/5}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0,664}{\sqrt{Re_x}}$	$\frac{\theta}{x} \simeq \frac{0,016}{(Re_x)^{1/7}}$	$\frac{\theta}{x} \simeq \frac{0,037}{(Re_x)^{1/5}}$
Local skin friction coefficient	$C_{f_x} = \frac{0,664}{\sqrt{Re_x}}$	$C_{f_x} \simeq \frac{0,027}{(Re_x)^{1/7}}$	$C_{f_x} \simeq \frac{0,059}{(Re_x)^{1/5}}$

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(a) Obtained from one-seventh power law

(b) Obtained from one-seventh power law combined with emperical data for turbulent flow through smooth pipe.

The one-seventh-power law is not the only turbulent boundary layer approximation used by fluid mechanics. Another is the Log Law, a semi-empirical expression that turns out to be valid not only for flat plate BL but also for fully developed turbulent pipe flow velocity profile.

The log-wall can be applied on all wall-bounded turbulent B.L.

The Log-Law is commonly expressed in variables nondimensionalized by characteristic velocity called the friction velocity, u_*

The Log-Law $\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{y u_*}{\nu}\right) + B$ (37)

Friction velocity $u_* = \sqrt{\frac{\tau_w}{\rho}}$ (38)

κ and B are constant $\kappa = 0,40$ to $0,41$ and $B = 5,0$ to 5.5. Near the wall does not work very well $\ln(0)$ is undifined.

Spalding's Law of the Wall (Valid to the all way to the wall)

friction velocity   velocity projile   constant

(39) $$\frac{y u_*}{\nu} = \frac{u}{u_*} + e^{-\kappa B}\left[e^{\kappa(u/u_*)} - 1 - \kappa\left(\frac{u}{u_*}\right) - \frac{\left[\kappa\left(\frac{u}{u_*}\right)\right]^2}{2} - \frac{\left[\kappa\left(\frac{u}{u_*}\right)\right]^3}{6}\right]$$

kinematic viscosity

Example.

Air at 20°C flows at $V = 10.0$ m/s over a smooth flat plate of length $L = 15,2$ m plot the Turbulent B.L. in physical variables (u as a function of y) at $x = L$, comparing the Power one-seventh, log law and Spadding's Law

$\nu = 1,516 \times 10^{-5}$ m²/s

$$Re_x = \frac{Vx}{\nu} = \frac{10 \cdot 15.2}{1,516 \times 10^{-5}} = 1,00 \times 10^{7}$$

$$\delta = \frac{0{,}16\,x}{(Re_x)^{1/7}} = 0{,}24\,m \qquad C_{fx} = \frac{0{,}027}{(Re_x)^{1/7}} = 2.70\times10^{-3}$$

Log Law: for friction velocity

$$u_* = \sqrt{\frac{\tau_w}{\rho}} = U\sqrt{\frac{C_{fx}}{2}} = 10.0\sqrt{\frac{2.7\times10^{-3}}{2}} = 0{,}367\,m/s$$

As $\tau_w = \frac{C_{fx}}{2}\rho U^2$

Log Law

$$\frac{u}{u_*} = \frac{1}{\kappa}\cdot \ln\left(\frac{y u_*}{\nu}\right) + B$$

$$y = \frac{\nu}{u_*}\cdot e^{\kappa\left(\frac{u}{u_*} - B\right)}$$

Kinematic viscos

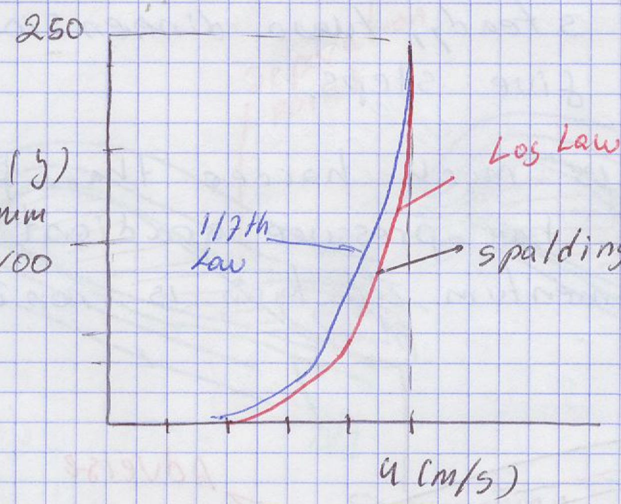
friction velocity

constant

Velocity profile.

Spalding's Law

On the wall, we have y as a function of u. Eq. (39).



Instead of a physical variable plot with linear axes as the above figure, semi-log plot of nondimensional variables is often drawn to magnify near wall region.

The most common notation in the B.L. literature for nondimensional variable is y^+ and u^+

Law of the wall variables.

$$y^+ = \frac{y u_*}{\nu} \qquad (40)$$

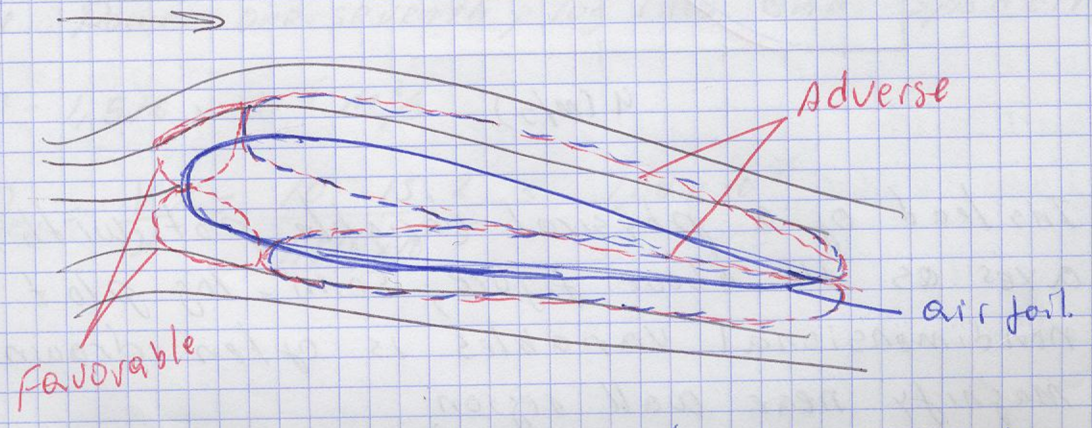
$$u^+ = \frac{u}{u_*}$$

Boundary Layers with Pressure Gradients

These includes:
- External Flow B.L. Problem
- internal Flow B.L. problem

We can use the flat plate boundary layers results as ballpark estimates for things as location of transition to turbulences, B.L. thickness, skin friction, etc. But when more accuracy is needed we need to solve B.L. equations for steady, two dimensional flow case using the five steps.

This analysis is much harder than ~~for~~ the flat plate since the pressure gradient term $\left(U \frac{dU}{dx}\right)$ in x-momentum equation is non zero.



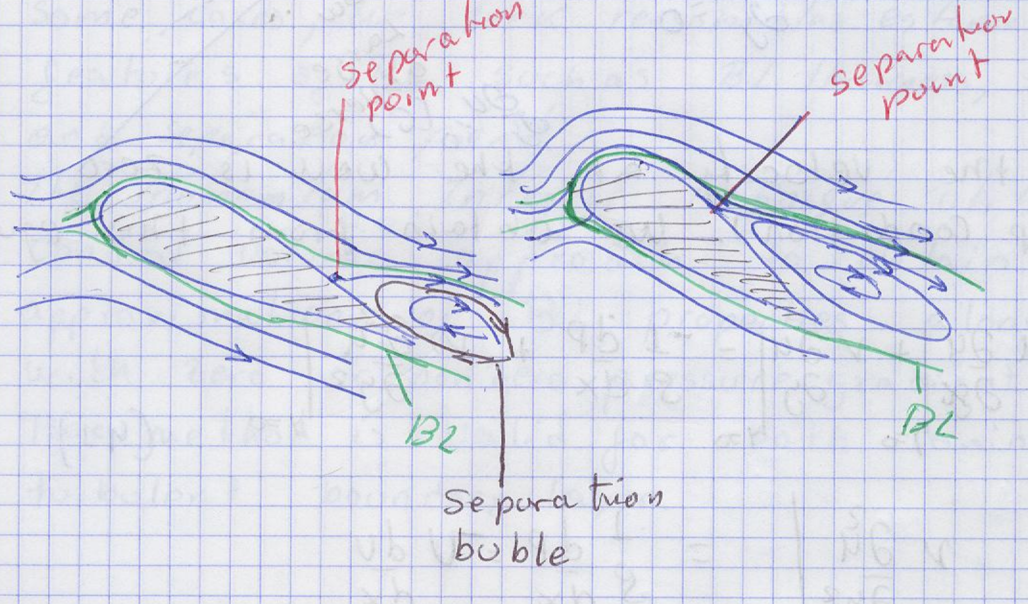
Favorable pressure gradient in the front portion and adverse pressure gradient in the rear portion of the body.

When the flow in the inviscid and/or irrotational outer flow region (outside boundary layer), accelerates $U(x)$ increases and $P(x)$ decreases. This is referedto the favorable pressure gradient. It is favorable because the BL. in such an accelerating flow is usually thin, hugs closely to the wall, and therefore is not likely to separate from the wall.

When the outer flow decelerates, $U(x)$ decreases, $P(x)$ increases, and we have unfavorable or adverse pressure gradient. At this condition the BL. is usually thicker, does not hug closely to the wall, and is much more likely to separate from the wall.

If the adverse pressure gradient is strong enough

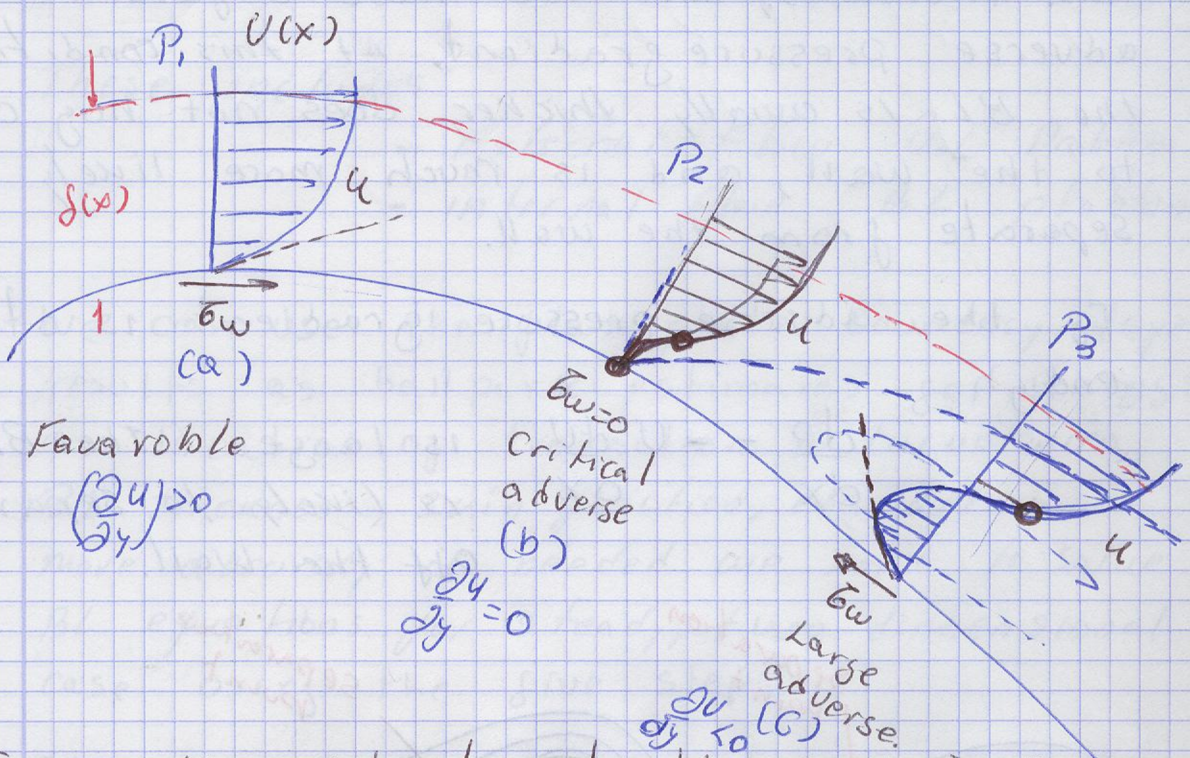
$$\frac{dP}{dx} = -U\frac{dU}{dx}$$ is large. The BL. is likely to separate off the wall.



Note: BL equations are not valid downstream of separation point because of reverse flow in the separation buble. No parabolic nature of the flow field and rendering the BL. equations inapplicable.

This case, the full Navier-stoke equations must be used in place of the B.L approximation

When the angle of attack is high, the separation point moves near the front of the airfoil; the separation bubble covers nearly the entire upper surface of the airfoil - this condition ~~this~~ is known as stall, this implies a loss of lift and a marked increase in aerodynamic drag.



Since the velocity at the wall is zero (no-slip condition). We obtain from this equation

$$u \frac{\partial u}{\partial x}\bigg|_{y=0} + v \frac{\partial u}{\partial y}\bigg|_{y=0} = -\frac{1}{\rho}\frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2}\bigg|_{y=0} \qquad (41)$$

$$\nu \frac{\partial^2 u}{\partial y^2}\bigg|_{y=0} = \frac{1}{\rho}\frac{dP}{dx} = -U \frac{dU}{dx}$$

Under favorable pressure gradient conditions (accelerating flow). $\frac{dU}{dx}$ is positive and second derivative of u at the wall is negative

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} < 0.$$

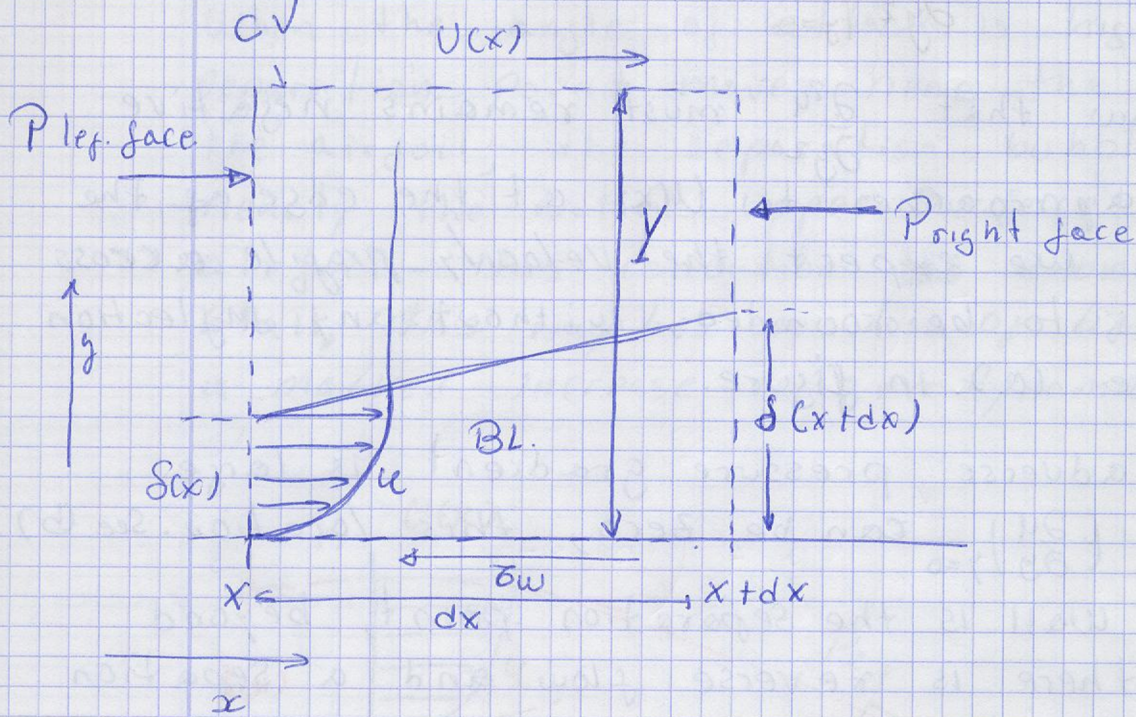
We know that $\frac{\partial^2 u}{\partial y^2}$ must remains negative as u approaches to $U(x)$ at the edge of the B.L. So we expect the velocity profile across the B.L. to be roonded, without any inflection Point. See (a). in figure.

If the adverse pressure gradient is large enough $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ can be zero, this location. See (b) along a wall is the separation point, beyond which there is reverse flow and a sepation bubble. See (c).

The Momentum Integral Technique for B.L

In many practical engineering applications, we do not need to know all the details inside BL, In some cases, we seek reasonables estimates of gross features of BL. such as B.L. thicuness, skin friction and separation point.

The momemtum Integral technique utilizes a control volume approach to obtain quantitative approximations of B.L. properties along surfaces with zero or nonzero pressure gradient as airfoils. This method is valid for both laminar and turbulent boundary layer.

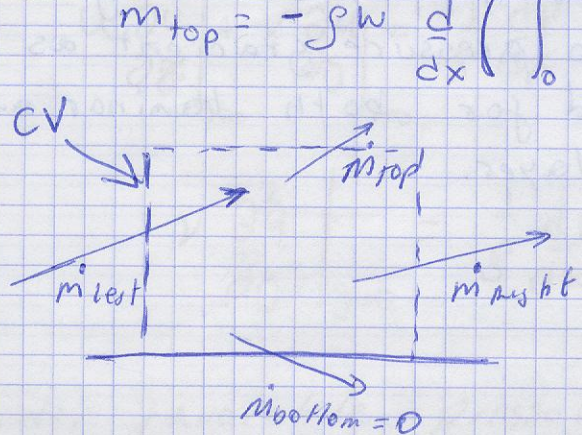


By BL approximation $\frac{\partial P}{\partial y} = 0$, at left face the pressure is P; at right the pressure is $P + \frac{dP}{dx}.dx$ mass flow rate through the left face as

$$\dot{m}_{left\,face} = \rho W \int_0^Y u\,dy \quad \text{and.}$$

(42) $$\dot{m}_{right\,face} = \rho W \left[\int_0^Y u\,dy + \frac{d}{dx}\left(\int_0^Y u\,dy \right) dx \right]$$

$$\dot{m}_{top} = -\rho W \frac{d}{dx}\left(\int_0^Y u\,dy \right) dx$$



x- momentum.

The net momentum flux out of the control volume must be balanced by the force due to the shear stress acting on the CV. by the wall and the net pressure force on the control surface

$$\underbrace{\sum F_{x\,body}}_{\text{Ignore gravity}}^{\cancel{\ }0} + \sum F_{x\,surface} = \int_{CS} (\rho \vec{V}) . \vec{V} . \vec{n}\, dA \qquad (43)$$

$$\sum F_{x\,surface} = \int_{Left} (\rho u) \vec{V} . \vec{n}\, dA + \int_{Right} (\rho u) \vec{V} . \vec{n}\, dA + \int_{TOP} (\rho u) \vec{V} . \vec{n}\, dA$$

$$\underbrace{Yw P - Yw \left(P + \frac{dP}{dx} . dx\right)}_{\text{Pressure gradiente}} - \underbrace{w\, dx\, \tau_w}_{\text{Shear stress}} = \underbrace{-\rho w \int_0^Y u^2 dy}_{\text{Left}} + \cancel{\ }$$

(44)

$$\rho w \left[\int_0^Y u^2 dy + \frac{d}{dx}\left(\int_0^Y u^2 dy \right) dx + \right.$$

$$\dot{m}_{TOP} . U$$

$$-Yw \frac{dP}{dx} - w \tau_w = \rho w \frac{d}{dx}\left(\int_0^Y u^2 dy \right) dx - \rho w U \frac{d}{dx}\left(\int_0^Y u\, dy \right) dx \quad (45)$$

We know at outer flow $-\rho U \frac{dU}{dx} = \frac{dP}{dx}$ and

$Y = \int_0^Y dy$ and dividing by ρ, we rewrite (45)

$$(46) \quad U \frac{dU}{dx} \int_0^Y dy - \frac{\tau_w}{\rho} = \frac{d}{dx} \left[\left(\int_0^Y u^2 dy \right) - U \frac{d}{dx} \left(\int_0^Y u\, dy \right) \right]$$

by utilizing product rule of differentiation in reverse

Product rule

$$\frac{d}{dx}\left(U\int_0^y u\,dy\right) = U\frac{d}{dx}\left(\int_0^y u\,dy\right) + \frac{dU}{dx}\cdot\int_0^y u\,dy$$

Product rule in reverse

$$U\frac{d}{dx}\int_0^y u\,dy = \frac{d}{dx}\left(U\int_0^y u\,dy\right) - \frac{dU}{dx}\int_0^y u\,dy$$

we write the (46)

$$\frac{d}{dx}\left(\int_0^y u(U-u)\,dy\right) + \frac{dU}{dx}\int_0^y (U-u)\,dy = \frac{\tau_w}{\rho} \quad (47)$$

we multiply and divide the first term by U^2

$$(48) \quad \frac{d}{dx}U^2\left(\int_0^y \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right) + U\frac{dU}{dx}\int_0^y\left(1-\frac{u}{U}\right)dy = \frac{\tau_w}{\rho}$$

If we integrate from 0 to ∞ we obtain

$$\frac{d}{dx}U^2\left(\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right) + U\frac{dU}{dx}\int_0^\infty\left(1-\frac{u}{U}\right)dy = \frac{\tau_w}{\rho} \quad (49)$$

$$\frac{d}{dx}(U^2\theta) + U\frac{dU}{dx}\cdot\delta^* = \frac{\tau_w}{\rho} \quad (50)$$

Equation (50) is called the Karmán integral equation. (Karmán was a student of Prandtl)

Another alternate form equation (50) is obtained by performing the product rule.

$$2U\frac{dU}{dx}\theta + \frac{d\theta}{dx}U^2 + U\frac{dU}{dx}.\delta^* = \frac{\tau_w}{\rho}$$ dividing by U^2

$$\frac{2}{U}\frac{dU}{dx}\theta + \frac{d\theta}{dx} + \frac{1}{U}\frac{dU}{dx}\delta^* = \frac{\tau_w}{\rho U^2}$$

$$\frac{dU}{dx}\left(\frac{2\theta}{U} + \frac{\delta^*}{U}\right) + \frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2}$$

$$\frac{\theta}{U}\frac{dU}{dx}\left(2 + H\right) + \frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} \quad (51)$$

$$\frac{C_{fx}}{2} = \frac{\tau_w}{\rho U^2}$$

(52) $H = \frac{\delta^*}{\theta}$ Shape factor.

$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho U^2}$ local skin friction coefficient.

Note that both H and C_{fx} are functions of x for general case of a boundary layer with non zero pressure gradient developing along a surface.

For a special case where $U(x) = U = \text{constant}$ we obtain $C_{fx} = 2\frac{d\theta}{dx}$ (53)

Example.

Suppose that we only know two things about the turbulent B.L. over a flat plate. the local skin friction coefficient.

$$C_{fx} = \frac{0.027}{(Re_x)^{1/7}}$$ and

B.L. profile shape (one-seventh powe law)

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \quad y \leq \delta \qquad \frac{u}{U} = 1 \quad \text{for } y > \delta$$

Estimate how δ, δ^* and θ vary with x

$$\theta = \int_0^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$\theta = \int_0^{\delta} \left(\frac{y}{\delta}\right)^{1/7}\left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = \frac{7}{72}\,\delta.$$

$$\delta^{*} = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = \frac{1}{8}\,\delta$$

for a flat plate with $U(x) = U = \text{constant}$

$$C_{f} = 2\frac{d\theta}{dx} = \frac{14}{72}\,\frac{d\delta}{dx}$$

$$\frac{d\delta}{dx} = \frac{72}{14}\,0{,}027\,(Re_x)^{-1/7}$$

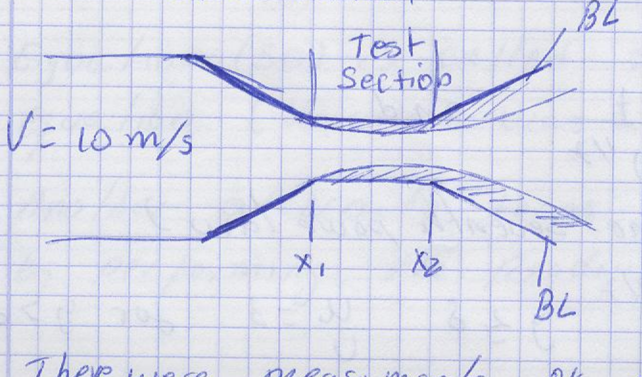
Boundary layer thickness $\quad \dfrac{\delta}{x} = \dfrac{0{,}16}{(Re_x)^{1/7}}$

Displacement thickness $\quad \dfrac{\delta^{*}}{x} = \dfrac{0{,}020}{(Re_x)^{1/7}}$

Momentum thickness $\quad \dfrac{\theta}{x} = \dfrac{0{,}016}{(Re_x)^{1/7}}$

Example

Wind tunnel



Air at 20°C and Atmospheric pressure
BL is fully turbulent at test section
Test section is 1.8 m long and 0.50m

There were measoments of $\delta_1 = 4{,}2\,cm \quad \delta_2 = 7{,}2\,cm$

were determined by power of eigth the BL. profile

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/8} \quad \text{for} \quad y \leq \delta \quad \text{and} \quad \frac{u}{U} = 1 \quad y \geq \delta$$

Estimate the total skin friction drag force F_D acting on the bottom wall of the wind tunnel test section

solution

$\rho = 1,204\ kg/m^3 \quad \nu = 1,516 \times 10^{-5}\ m^2/s$ at (air) 20°C

momentum thickness

$$\theta = \int_0^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^{\delta} \left(\frac{y}{\delta}\right)^{1/8}\left[1 - \left(\frac{y}{\delta}\right)^{1/8}\right] dy = \frac{4}{45}\delta$$

The Karman integral equation reduces to

$$C_{fx} = 2\frac{d\theta}{dx} \quad \text{flat plate} \quad U(x) = U = \text{constant}$$

in terms of shear stress.

$$\tau_w = \frac{1}{2}\rho U^2 C_{fx} = \rho U^2 \frac{d\theta}{dx} \quad \text{and}$$

$$F_D = w\int_{x_1}^{x_2} \tau_w\, dx = w\rho U^2 \int_{x_1}^{x_2} \frac{d\theta}{dx} \cdot dx$$

$$F_D = w\rho U^2 (\theta_2 - \theta_1)$$

$$F_D = w\rho U^2 \left(\frac{4}{45}\delta_2 - \frac{4}{45}\delta_1\right)$$

$$F_D = w\rho U^2 \frac{4}{45}(\delta_2 - \delta_1)$$

$$F_D = (0.5m) \cdot 1,204\frac{kg}{m^3} \cdot (10)^2\frac{m^2}{s^2} \cdot \frac{4}{45}(0,077 - 0,042)\, m\, \frac{s^2 N}{kg \cdot m}$$

$$F_D = 0,19\, N$$